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The equation $A \otimes x = B \otimes y$ over $(\max, +)$

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Abstract

For the two-sided homogeneous linear equation system $A \otimes x = B \otimes y$ over $(\max, +)$, with no infinite rows or columns in A or B, an algorithm is presented which converges to a finite solution from any finite starting point whenever a finite solution exists. If the finite elements of A, B are all integers, convergence is in a finite number of steps, for which a precise bound can be calculated if moreover one of A, B has only finite elements. The algorithm is thus pseudopolynomial in complexity. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

It is well-known that the structure of many discrete-event dynamic systems may be represented by square matrices A over the semiring

$$\mathfrak{R} = (\{-\infty\} \cup \mathbb{R}, \oplus, \otimes) = (\{-\infty\} \cup \mathbb{R}, \max, +).$$

If the initial event-times of such a system are represented by a vector s, then the event-times after r stages are given by the rth term of the *orbit*

$$\{A^{(r)} \otimes s \ (r = 1, 2, \ldots)\}, \quad \text{where } A^{(r)} = A \otimes A \otimes \cdots \otimes A \ (r\text{-fold}).$$

The *reachability problem* asks whether s can be chosen so that the orbit contains a given vector b. Clearly, the answer is affirmative if and only if event-times b can be achieved after one stage from suitable previous event-times, so algebraically the reachability problem produces the linear-equations problem: to solve $A \otimes x = b$. Some necessary facts relevant to this are reviewed in the next section.

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The principal topic we shall address arises from the *synchronisation problem*: can two different systems be set in motion so as subsequently to achieve the same event-times? Clearly, this asks whether x, y can be found to satisfy the equation

$$A \otimes x = B \otimes y \tag{1.1}$$

for given A, B.

The use of $-\infty$ for a matrix element a_{ij} is generally to model the fact that the system has no forward coupling from component j to component i. To avoid triviality, therefore, we shall assume throughout that A, B have at least one finite element on each row and on each column: such matrices are called *doubly G-astic* [2]. And since our vectors x, y, b, etc. represent times of physical events, we shall restrict our attention to situations where these are finite. These assumptions are algebraically self-consistent: all sums and products of doubly G-astic matrices are doubly G-astic [2]. In particular, with x finite and A doubly G-astic, all elements of the orbit are automatically finite.

In summary, therefore, we shall investigate finite solutions to (1.1) with A, B doubly G-astic.

A pair (x, y) satisfying (1.1) will now be called simply a *solution*. In Section 3, we present a straight-forward algorithm which converges to a solution in pseudopolynomial time from any finite initial pair whenever a solution exists.

Although the above motivation has assumed A, B to be square, there is no extra algebraic or algorithmic cost in assuming only that A, B have equal number of rows. This we shall do.

A related inhomogeneous equation in x only:

$$A \otimes x \oplus a = B \otimes x \oplus b \tag{1.2}$$

has received some attention in the literature and generates relatively complex analysis (see [2,4]) and references in [1]. It is of interest, therefore that instances of (1.2) can be reformulated as instances of (1.1). We consider this further in Section 10.

2. Background assumptions

We assume familiarity with the basic properties of the semiring $\mathfrak{R}$ and of matrix algebra over $\mathfrak{R}$, as set out in e.g. [1,3,5]. In particular, we shall make extensive use of the isotonicity of the scalar and matrix operations relative to the natural partial order. To avoid repetitive dimensioning statements, we assume all matrices conformable for the indicated operations.

For a given square matrix X over $\mathfrak{R}$, if there exist finite vector e, and $\lambda \in \mathbb{R}$, constituting an *eigenvector* and *eigenvalue* for X:

$$X \otimes e = \lambda \otimes e,$$

we say that X has *finitely soluble eigenproblem*. A sufficient condition is the finiteness of X; necessary and sufficient conditions are discussed further in [3]. λ is then [3] a unique function $\lambda(X)$ (the maximum cycle-mean) of the elements of X. Both λ and the generators of the space of eigenvectors can be determined in low-order polynomial time [1,3]. It is easy to see that if A, B are square and have a finite eigenvector in common, then a solution to (1.1) exists, which is readily found using these algorithms.

The system $\mathfrak{R}$ is embeddable in the self–dual system

$$\mathfrak{I} = (\{-\infty\} \cup \mathbb{R} \cup \{+\infty\}, \oplus, \otimes, \oplus', \otimes')$$

$$= (\{-\infty\} \cup \mathbb{R} \cup \{+\infty\}, \max, +, \min, +),$$

where the commutative operations $\otimes, \otimes'$ differ only in that

$$-\infty \otimes +\infty = -\infty, \quad -\infty \otimes' +\infty = +\infty.$$

For any matrix $A = [a_{ij}]$ over $\mathfrak{I}$, the *conjugate matrix* is $A^* = [-a_{ji}]$ obtained by negation and transposition. The following relations (2.1), (2.2), (2.3) hold for any matrices U, V, W over $\mathfrak{I}$:

$$(U \otimes' V) \otimes W \leqslant U \otimes' (V \otimes W), \tag{2.1}$$

$$U \otimes (U^* \otimes' W) \leqslant W, \tag{2.2}$$

$$U \otimes (U^* \otimes' (U \otimes W)) = U \otimes W. \tag{2.3}$$

A set of linear inequalities $A \otimes x \leqslant b$ over $\mathfrak{R}$ always possesses a solution. The greatest is

$$x = A^* \otimes' b, \tag{2.4}$$

which is finite for A doubly G-astic and b finite.

This *principal solution* is calculated in $\mathfrak{I}$ but lies in $\mathfrak{R}$, being finite. It is also the greatest solution of the following linear-equation system (2.5), if and only if any solution exists:

$$A \otimes x = b. \tag{2.5}$$

3. The alternating method

Eqs. (2.4) and (2.5) motivate the following *Alternating Method* for solving (1.1):

Initialise
  Choose arbitrary finite vector x
  Set $r = 0$; $x(0) = x$
Repeat
  Set $y =$ principal solution of $B \otimes y \leqslant A \otimes x$; $y(r) = y$
  Set $x =$ principal solution of $A \otimes x \leqslant B \otimes y$; $x(r+1) = x$
  Set $r = r + 1$
Until convergence
End

Define the maps

$$\pi:\ y \to A^* \otimes' (B \otimes y), \quad \psi: x \to B^* \otimes' (A \otimes x). \tag{3.1}$$

These are compositions of continuous isotone operations, and therefore are continuous and isotone.

The Alternating Method thus generates the pair-sequence $\{(x(r), y(r))\ (r = 0, 1, \ldots)\}$, where

$$x(r+1) = \pi(y(r)), \quad y(r) = \psi(x(r)). \tag{3.2}$$

We show in Section 7 that the pair-sequence $\{(x(r), y(r))\}$ converges to a solution if one exists. It is clear that each step of the algorithm has polynomial complexity, and in Section 10 we show that the algorithm as a whole has pseudopolynomial complexity.

4. An example

Suppose

$$A = \begin{bmatrix} 3 & -\infty & 0 \\ 1 & 1 & 0 \\ -\infty & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 2 \\ 3 & 1 \end{bmatrix}, \quad \text{so } A^* = \begin{bmatrix} -3 & -1 & +\infty \\ +\infty & -1 & -1 \\ 0 & 0 & -2 \end{bmatrix},$$

$$B^* = \begin{bmatrix} -1 & -3 & -3 \\ -1 & -2 & -1 \end{bmatrix}.$$

Set

$$x = x(0) = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}, \quad \text{say.}$$

The algorithm finds sequentially:

$$r = 0: \qquad A \otimes x = \begin{bmatrix} 8 \\ 6 \\ 4 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad B \otimes y = \begin{bmatrix} 4 \\ 5 \\ 4 \end{bmatrix},$$

$$r = 1:\ x = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad A \otimes x = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad B \otimes y = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix},$$

$$r = 2:\ x = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}, \quad A \otimes x = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}.$$

At this point, $A \otimes x(2) = B \otimes y(1)$, giving the solution $(x(2), y(1))$.

5. Stable solutions

We shall say that (x, y) is *stable* if

$$(x, y) = (\pi(y), \psi(x)) \tag{5.1}$$

If (x, y) is also a solution, we speak of a *stable solution*.

Theorem 5.1. *Any stable pair is a stable solution.*

Proof. If (x, y) is stable, then, using (2.2)

$$A \otimes x = A \otimes \pi(y) = A \otimes (A^* \otimes' (B \otimes y)) \leqslant B \otimes y = B \otimes \psi(x)$$

$$= B \otimes (B^* \otimes' (A \otimes x)) \leqslant A \otimes x.$$

Hence all these terms are equal and $A \otimes x = B \otimes y$. □

Theorem 5.2. *If (x, y) is a solution, then $(\pi(y), \psi(x))$ is a stable solution.*

Proof. We use (2.3) and the fact that (x, y) is a solution.

$$\psi(\pi(y)) = B^* \otimes' (A \otimes (A^* \otimes' (B \otimes y))) = B^* \otimes' (A \otimes (A^* \otimes' (A \otimes x)))$$
$$= B^* \otimes' (A \otimes x) = \psi(x).$$

Similarly, $\pi(\psi(x)) = \pi(y)$, whence $(\pi(y), \psi(x))$ is stable and therefore a solution. □

6. Properties of the sequence

Theorem 6.1. *The sequence $\{A \otimes x(r)\ (r = 0, 1, \ldots)\}$ is non-increasing.*

Proof. Applying standard inequality (2.2) to recurrences (3.2),

$$A \otimes x(r+1) \leqslant B \otimes y(r) \leqslant A \otimes x(r). \quad \square$$

Theorem 6.2. *The sequence $\{x(r)\ (r = 1, 2, \ldots)\}$ is non-increasing.*

Proof.

$$x(r+1) = \pi(y(r)) = \pi(B^* \otimes' (A \otimes x(r))).$$

So $x(r+1)$ is an isotone function of the non-increasing $A \otimes x(r)$. □

Theorem 6.3. *If a solution exists, then the sequence $\{x(r)\ (r = 1, 2, \ldots)\}$ is lower-bounded for any $x(0)$.*

Proof. For any stable solution (x, y), and $\mu \in \mathbb{R}$, it is immediate that $\mu \otimes (x, y)$ is again a stable solution, and μ may be chosen small enough so that $\mu \otimes x \leqslant x(1)$. Hence if a solution exists, then, using Theorem 5.2, a stable solution (u, v) exists such that $x(1) \geqslant u$. And if $x(r) \geqslant u$ for some r, then using (3.2) and isotonicity, we have

$$x(r+1) = (\pi \circ \psi)(x(r)) \geqslant (\pi \circ \psi)(u) = \pi(v) = u$$

and the result follows by induction. □

We remark that in the proof of Theorem 6.3, μ may in fact be chosen so that $\mu \otimes x \leqslant x(1)$, but with equality in at least one component. Theorems 6.2 and 6.3 then show that that component of $x(r)$ remains fixed in value for $r \geqslant 1$. Moreover, it is clear that analogues of Theorems 6.1–6.3 are provable for the sequence $\{y(r)\}$. Hence:

Theorem 6.4. *If all components of $x(r)$ or $y(r)$ have properly decreased after a number of steps, the algorithm may be halted with the conclusion that no solution exists.* □

7. Monotonic convergence

Theorem 7.1. *The pair-sequence $\{(x(r), y(r))\ (r = 0, 1, \ldots)\}$, generated by the alternating method, converges if and only if a solution exists. Convergence is then monotonic, to a stable solution, for any choice of $x(0)$.*

Proof. If $(x(r), y(r)) \to (\xi, \eta)$, then by continuity

$$(\xi, \eta) = \lim(x(r+1), y(r)) = \lim(\pi(y(r)), \psi(x(r))) = (\pi(\eta), \psi(\xi)).$$

Hence (ξ, η) is stable. Conversely, if a solution exists, the monotonic convergence of $\{x(r)\}$ follows from Theorems 6.2 and 6.3, and that of $\{y(r)\} = \{\psi(x(r))\}$ by isotonicity and continuity. □

By replacing $\mathbb{R}$ by one or other of its subgroups (under arithmetical addition), we obtain subsemirings of $\mathfrak{R}$: in particular, we may take the rationals or the integers. To avoid unnecessary notation, we shall then refer simply to *the rational case* or *the integer case*, as distinct from *the general case* when $\mathbb{R}$ itself is taken. Since the rationals are dense in the reals, and the numbers are usually supposed to refer to measurable physical amounts, it is typically the rational case which is relevant. For the Alternating Method, however, we may then regard the arithmetic as set in the domain of integer multiples of δ^{-1}, where δ is the LCM of the denominators of all finite elements of A, B and $x(0)$. The integer case is thus of central importance.

Clearly, a lower-bounded non-increasing integer sequence converges in a finite number of steps, whence:

Theorem 7.2. *In the integer case, if a solution exists, the Alternating Method produces a solution in a finite number of steps.* □

Such finite termination is illustrated by the example of Section 4.

Consider an instance of (1.1) in the integer case. This may also be considered as an instance in the general case and may have a non-integer solution. However, even in the general case, $x(0)$ may always be chosen with integer elements and it is clear that all pairs of the sequence $\{(x(r), y(r))\}$ will then have integer elements and so therefore will any limit. Thus the following holds.

Theorem 7.3. *If an instance of* (1.1) *in the integer case has a solution when viewed as an instance in the general case*, *then it possesses a solution in the integer case*, *and conversely.* □

8. Convergence speed in the finite-integer case

A bound may be calculated on the number of steps to convergence in the integer case, if one of the matrices A, B has only finite elements. Assume first that A is finite.

Suppose $x(1) = \gamma$ and that a solution exists. For convenience, write x for $x(r)$. From the remark following Theorem 6.3, we know that at least one component of x never falls in value—let us call such a component a *sleeper*. Suppose the jth component $x_j = \gamma_j$ is a sleeper and the kth component x_k is a non-sleeper. Now, x_k plays no part in the evaluation of $A \otimes x$ if

$$a_{ik} + x_k < a_{ij} + \gamma_j, \quad \forall i,$$

that is, if x_k takes a value below u_{kj}, where

$$u_{kj} = \min_i (a_{ij} - a_{ik} + \gamma_j).$$

This last expression is just $(A^* \otimes' A)_{kj} + \gamma_j$. If $x_k < u_{kj}$, we say that x_k is *dominated* by the sleeper γ_j. Since x_k is non-increasing, the domination persists in subsequent iterations. Now, *some* component is a sleeper, so x_k is certainly dominated if it falls in value below β_k, where

$$\beta_k = \min_j u_{kj},$$

wherein index j is now unrestricted. This last expression is then just

$$\min_j ((A^* \otimes' A)_{kj} + \gamma_j) = (A^* \otimes' A \otimes' \gamma)_k, \tag{8.1}$$

calculated by pre-computing $\beta = A^* \otimes' A \otimes' \gamma$. So the fall of x_k sufficient for x_k to be permanently dominated does not exceed

$$w_k = \gamma_k - \beta_k + 1. \tag{8.2}$$

Notice that the finiteness of A guarantees our obtaining finite values for the u_{kj} and hence for β_k and w_k. If the dimension of x is n, there are at most $(n-1)$ non-sleepers, and at each step prior to convergence, at least one not-yet-dominated non-sleeper falls by at least unity, because at least one component of $A \otimes x$ must fall. Hence the number of steps to convergence does not exceed the sum of the greatest $(n-1)$ of the w_k, which yields a termination criterion for the algorithm: if convergence has not occurred in this number of steps, it is certain that no solution exists.

Theorem 8.1. *The number of steps to convergence in the integer case if A is finite does not exceed*

$$(n-1) * (1 + \gamma^* \otimes A^* \otimes A \otimes \gamma),$$

where n is the column-dimension of A, and $x(0) = \gamma$.

Proof. An overestimate of the sum of the greatest $(n-1)$ of the w_k is

$$(n-1) * \max_k w_k = (n-1) * \max_k(\gamma_k - \beta_k + 1) \quad \text{(using (8.2))}$$

$$= (n-1) * (1 + \beta^* \otimes \gamma) = (n-1) * (1 + \gamma^* \otimes A^* \otimes A \otimes \gamma)$$

$$\text{(using (8.1))}. \qquad \square$$

Theorem 8.2. *For given square matrix $D = [d_{ij}]$ with finitely soluble eigenproblem, the minimum of $x^* \otimes D \otimes x$ w.r.t. x equals the eigenvalue $\lambda(D)$, with minimiser x equal to any finite eigenvector of D.*

Proof. By taking x equal to any finite eigenvector, the achieved value is

$$x^* \otimes D \otimes x = x^* \otimes (\lambda(D) \otimes x) = \lambda(D) \otimes (x^* \otimes x) = \lambda(D).$$

Now, in [3, Theorem 25-10], it is shown that $\lambda(D)$ gives the optimal value of λ in the following linear-programming problem in variables $\lambda, x_1, \ldots, x_n$:

$$\text{minimise } \lambda, \text{ subject to } \lambda + x_i - x_j \geqslant d_{ij} \quad \text{for all finite } d_{ij}.$$

The constraints are equivalent to

$$\lambda \geqslant \max_{ij} (-x_i + d_{ij} + x_j) = x^* \otimes D \otimes x,$$

in fact to

$$\lambda = x^* \otimes D \otimes x,$$

since λ is being minimised, so the minimum value of $x^* \otimes D \otimes x$ is $\lambda(D)$. $\square$

Theorems 8.1 and 8.2 give a plausible basis for taking an eigenvector of $A^* \otimes A$ as a starting solution in the Alternating Method.

It is clear that we could argue in terms of the sequence $\{y(r)\}$ instead of $\{x(r)\}$, obtaining an analogous bound if B is finite, and taking the smaller bound if both A, B are finite. Hence:

Theorem 8.3. *The alternating method has pseudopolynomial complexity.* $\square$

9. A sufficient condition

Define

$$M = (A^* \otimes' B) \otimes (B^* \otimes' A).$$

Theorem 9.1. *If M has finitely soluble eigenproblem, then $\lambda(M) \leqslant 0$.*

Proof. If ξ is a finite eigenvector of M then, writing λ for $\lambda(M)$, and using (2.1), (2.2) and isotonicity,

$$\begin{aligned}\lambda \otimes (A \otimes \xi) &= A \otimes (\lambda \otimes \xi) = A \otimes M \otimes \xi = A \otimes (A^* \otimes' B) \otimes (B^* \otimes' A) \otimes \xi \\ &\leqslant B \otimes (B^* \otimes' A) \otimes \xi \leqslant B \otimes (B^* \otimes' (A \otimes \xi)) \leqslant A \otimes \xi \qquad (9.1)\end{aligned}$$

implying the result. □

Theorem 9.2. *If $\lambda(M) = 0$, then $(\xi, \psi(\xi))$ is a solution for every finite eigenvector ξ of M.*

Proof. If $\lambda = 0$, then all the terms in (9.1) are equal, and

$$A \otimes \xi = B \otimes (B^* \otimes' (A \otimes \xi)) = B \otimes \psi(\xi). \qquad \square$$

Since the eigenvalue, and the space of eigenvectors, of any square matrix can be determined by standard algorithms of low-order polynomial complexity, this allows efficient determination of a solution when $\lambda(M) = 0$. Unfortunately, this condition is not necessary for the existence of a solution, as the example of Section 4 shows. Here, M may be calculated as

$$\begin{bmatrix} -\infty & -\infty & -4 \\ -\infty & -\infty & -1 \\ -\infty & -\infty & -2 \end{bmatrix}$$

and then $\lambda(M)$ is evaluated [1] at -2.

10. The inhomogeneous case

In view of the relatively straightforward nature of the alternating method, it is of interest to note that it may be used to seek finite solutions to instances of the inhomogeneous problem (1.2) with A, B doubly G-astic and a, b finite. First, consider

$$C \otimes x = D \otimes x. \qquad (10.1)$$

This is equivalent to

$$C \otimes x = y, \quad D \otimes x = y$$

and hence to

$$\begin{bmatrix} C \\ D \end{bmatrix} \otimes x = \begin{bmatrix} I \\ I \end{bmatrix} \otimes y, \tag{10.2}$$

where I is the usual identity matrix over $\mathfrak{R}$, with diagonal elements zero and off-diagonal elements equal to $-\infty$. Clearly (10.2), is an instance of (1.1). Since I is doubly G-astic, we conclude that a solution to any instance of (10.1) with C, D doubly G-astic may be found by the Alternating Method whenever a solution exists.

Now introduce an extra single scalar variable z, and consider

$$A \otimes x \oplus a \otimes z = B \otimes x \oplus b \otimes z. \tag{10.3}$$

Any instance of (10.3) with A, B doubly G-astic and a, b finite is an instance of (10.1) with C, D doubly G-astic, on substituting C by $[A, a]$; D by $[B, b]$; x by $[\begin{smallmatrix} x \\ z \end{smallmatrix}]$. But it is clear that (10.3) has a solution with z finite if and only if it has a solution with $z = 0$, since if $[\begin{smallmatrix} x \\ z \end{smallmatrix}]$ is a solution then so is $(-z) \otimes [\begin{smallmatrix} x \\ z \end{smallmatrix}]$. And the x-parts of finite solutions to (10.3) with $z = 0$ are precisely the finite solutions to (1.2).

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